CS 58500 Theoretical Computer Science Toolkit (Spring 2026) Problem Set # 3
{{Your name}} **Due: Apr 17th**

The questions have been labeled with the date of the lecture in which the relevant material is covered.

Problem 1. Concentration of Trajectories (03/26). (30 pts)

The goal of this problem is to prove the following theorem: Let P be an ergodic Markov chain on a finite state space Ω which is reversible w.r.t. a probability measure μ. Let $(X_t)_{t\geq 0}^{\infty}$ be a trajectory of P with the initial state X_0 drawn from the stationary distribution μ. Then for every $T \geq 1$, every $\epsilon > 0$, and every 1-bounded function $f : \Omega \to [0,1]$, we have the concentration inequality

$$\Pr\left[\left|\frac{1}{T}\sum_{t=0}^{T-1} f(X_t) - \mathbb{E}_\mu[f]\right| \geq \lambda_2^*(\mathsf{P}) + \epsilon\right] \leq 2\exp(-C\epsilon^2 T)$$

for some universal constant $C > 0$, where $\lambda_2^*(\mathsf{P}) := \max_{i>1}\{|\lambda_i(\mathsf{P})|\}$ gives the second largest eigenvalue of P in absolute value.

1. (15 pts) Like most proofs of Chernoff-like concentration inequalities, the crucial step is obtaining a good bound on the moment generating function. Let $\theta \geq 0$ be a parameter to be determined later. Establish the following bound:

   $$\mathbb{E}\left[\exp\left(\theta\sum_{t=0}^{T-1} f(X_t)\right)\right] \leq \|M_f\|^{T-1} \cdot \mathbb{E}_\mu[e^{\theta f}]\,,$$

   where $\mathbf{M}_f \in \mathbb{R}^{\Omega\times\Omega}$ is defined as:

   $$\mathbf{M}_f := \mathrm{diag}(e^{\theta f/2}) \cdot \mathrm{diag}(\mu)^{1/2}\mathsf{P}\mathrm{diag}(\mu)^{-1/2} \cdot \mathrm{diag}(e^{\theta f/2})$$

2. (15 pts) Complete the proof of the theorem.
   (Hint: Aim for the upper bound $\|\mathbf{M}_f\| \leq \lambda_2^*(\mathsf{P}) \cdot e^\theta + (1 - \lambda_2^*(\mathsf{P})) \cdot \mathbb{E}_\mu[e^{\theta f}]$. It may also be helpful to use the inequality $1 - e^{-x} \geq x - \frac{1}{2}x^2$, which holds for all $x \geq 0$.)

Solution.

Problem 2. Matrix Bernstein (04/02). (50 pts)

1. (20 pts) Prove the following matrix Bernstein's inequality:

   > Consider a finite sequence $\{\mathbf{Z}_k\}$ of independent, random matrices with common dimension $d_1 \times d_2$. Assume that $\mathbb{E}[\mathbf{Z}_k] = \mathbf{0}$ and $\|\mathbf{Z}_k\| \leq L$ for each index k. Let $\mathbf{Z} := \sum_k \mathbf{Z}_k$. Let $\nu(\mathbf{Z})$ be the matrix variance proxy of the sum:
   >
   > $$\nu(\mathbf{Z}) := \max\{\|\mathbb{E}[\mathbf{Z}\mathbf{Z}^\top]\|, \|\mathbb{E}[\mathbf{Z}^\top\mathbf{Z}]\|\}\,.$$
   >
   > Then,
   >
   > $$\Pr[\|\mathbf{Z}\| \geq t] \leq (d_1 + d_2) \cdot \exp\left(-\frac{t^2/2}{\nu(\mathbf{Z}) + Lt/3}\right) \quad \forall t \geq 0\,.$$

   You can directly use the Subadditivity of matrix MGF.

2. (30 pts) Prove the expectation bound in the same setting:

   $$\mathbb{E}[\|\mathbf{Z}\|] \leq \sqrt{2\nu(\mathbf{Z})\log(d_1 + d_2)} + \frac{1}{3}L\log(d_1 + d_2)\,.$$

Solution.

Problem 3. Differential Operator (04/07). (30 pts)

Given $x \in \{0,1\}^n$, we represent $x|_{i=1}$ as the bit-string identical to x except that its i-th coordinate is fixed to 1. Similarly, $x|_{i=0}$ is the bit-string that is identical to x except that its i-th coordinate is fixed to 0. For any Boolean function $f : \{0,1\}^n \to \mathbb{R}$, let $D_i f$ be the function $\{0,1\}^n \to \mathbb{R}$ defined as follows

$$D_i f(x) := f(x|_{i=1}) - f(x|_{i=0})\,.$$

Express the Fourier coefficients $\widehat{D_i f}$ as a function of $\widehat{f}$.
Solution.

Problem 4. Fourier Transformation Matrix (40 pts)

For functions $\{0,1\}^n \to \mathbb{R}$, we defined the basis functions as follows. For all $S, x \in \{0,1\}^n$, we defined $\chi_S(x) := (-1)^{\langle S,x\rangle} = (-1)^{S_1x_1+\cdots+S_nx_n}$. Given this definition of the Fourier basis functions, the definition of the Fourier transformation matrix (or Walsh matrix) $\mathcal{F}_n \in \{1/N, -1/N\}^{N\times N}$, where $N = 2^n$, is as follows:

$$(\mathcal{F}_n)_{i,j} = \frac{1}{N}\chi_j(i) \quad \forall i, j \in \{0,1\}^n\,.$$

On the other hand, the Fourier transformation matrix can be defined using the Kronecker product. Let $\mathbf{A} \in \mathbb{R}^{a\times b}$ and $\mathbf{B} \in \mathbb{R}^{a'\times b'}$ be two matrices. We define the block matrix $\mathbf{C} = \mathbf{A} \otimes \mathbf{B} \in \mathbb{R}^{aa'\times bb'}$ as follows:

$$\mathbf{C}_{i,j} = a_{i,j}\mathbf{B} \in \mathbb{R}^{a'\times b'} \quad \forall i \in [a], j \in [b]\,.$$

The Fourier transformation matrix is constructed as follows:

Base case. Define

$$\mathcal{G}_1 := \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Recursive construction. Define, for $n > 1$,

$$\mathcal{G}_n := \mathcal{G}_1 \otimes \mathcal{G}_{n-1}$$

1. (20 pts) Prove, by induction, that $\mathcal{F}_n = \mathcal{G}_n$.
2. (10 pts) Prove that $\mathcal{F}_n^2 = \frac{1}{N}\mathbf{I}$
3. (10 pts) Show that the Mat-Vec product $\mathcal{F}_n v$ for any $v \in \mathbb{R}^N$ can be done in $\mathcal{O}(N \log N)$ time.

Solution.